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STUDY OF CRACK OPENING USING THE WEIGHTING FUNCTIONS METHOD*

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Some results of calculations of the opening of rectilinear, disc-like cracks under the action of a given system of forces, are given in /1-3/. A study of the opening of internal and surface cracks of more complex form is of interest, since in a number of cases it enables one to determine the depth of the crack from its known opening at the surface.

Formulas are obtained for the opening of elliptical, internal or surface cracks which occur when the body is acted upon by an arbitrary static load symmetrical about the plane of the crack.

1. Let us consider an elastic body with a rectilinear skew crack $0 \leqslant x \leqslant l$, internal or emerging at the surface $x = 0$. A weighting functions (WF) method was proposed in /4/ for computing the stress intensity factors (SIF) at the crack tip, and the possibility of using the method to determine the displacement field was suggested. When the elastic deformation energy $W(l)$ and the displacement of the upper edge of the crack $v(x, l)$ are both known for a certain external load, the WF can be found using the formula /4/

$$h(x, l) = \frac{1}{2}\left(\frac{1}{E'}\frac{\partial W}{\partial l}\right)^{-1/2}\frac{\partial v(x, l)}{\partial l} \tag{1.1}$$

where $E' = E/(1-\nu^2)$ for plane deformation, $E' = E$ for the state of plane stress, E is the modulus of elasticity, ν is Poisson's ratio and $h(x, l)$ is independent of the type of loading.

We have the following formula for the SIF $K(l)$ at the tip $x = l$:

$$K(l) = 2\int_0^l \sigma(x) h(x, l)\, dx \tag{1.2}$$

*Prikl.Matem.Mekhan., 51, 1, 140-145, 1987 where $\sigma(x)$ denotes the arbitrary distribution of stresses applied to the crack edges. We also know that

$$K^2(l) = E'\partial W/\partial l \tag{1.3}$$

This enables us to rewrite (1.1) in the form

$$h(x,\ l) = {}^1/_2\,(E'/K(l))\,\partial v(x,\ l)/\partial l \tag{1.4}$$

When the crack is internal, we have $v(0,\ l) = 0$, and this leads, according to (1.4), to $h(0,\ l) = 0$.

Thus, as was shown in /4/, when $K(l)$ and $v(x,\ l)$ are known, then no matter what the external load, formulas (1.2) and (1.4) will enable us to determine $K(l)$ for any function $\sigma(x)$.

Let us now turn our attention to the problem of determining the displacement of the crack edge, and obtain $v(x,\ l)$ from (1.4), taking into account the fact that $v(l,\ l) = 0$. We will have

$$v(x,l) = \frac{2}{E'}\int_x^l K(t)\,h(x,t)\,dt \tag{1.5}$$

Formulas (1.2) and (1.5) enable us to determine the opening of a crack when $\sigma(x)$, is arbitrary, provided that the WF $h(x,\ l)$ is known.

We note the following. Let us assume that concentrated unit forces are applied symmetrically to the upper and lower edge of the crack, at a distance ζ from the origin of coordinates, i.e. $\sigma(x) = \delta(x-\zeta)$, where $\delta(x-\zeta)$ is the Dirac function. Denoting by $K_0(\zeta,\ l)$ the SIF at the tip $x = l$ corresponding to the concentrated unit forces, we obtain from (1.2) the relation $K_0(\zeta,\ l) = 2h(\zeta,\ l)$ whose substitution into (1.5) yields the formula

$$v(x,l) = \frac{1}{E'}\int_x^l K(t)\,K_0(x,t)\,dt \tag{1.6}$$

obtained earlier by Paris /1/ by a different method. Formulas (1.2) and (1.5) represent a specific example of formula (1.6) in the case when $K(l)$ and $K_0(\zeta,\ l)$ are determined using the weighting functions method. However, the derivation of (1.5) using the above method is of independent interest, as it can be generalized to the case of a crack of more complex form.

Substituting (1.2) into (1.5) we obtain

$$v(x,l) = \frac{4}{E'}\int_x^l \left[h(x,t)\int_0^t \sigma(\xi)\,h(\xi,t)\,d\xi\right]dt = \tag{1.7}$$

$$\frac{4}{E'}\left\{\int_0^x \left[\sigma(\xi)\int_x^l h(x,t)\,h(\xi,t)\,dt\right]d\xi + \int_x^l \left[\sigma(\xi)\int_\xi^l h(x,t)\,h(\xi,t)\,dt\right]d\xi\right\}$$

and this yields

$$v(x,l) = \frac{4}{E'}\int_0^l \sigma(\xi)\,f(x,\xi,l)\,d\xi,\quad f(x,\xi,t) = \int_{\max(x,\xi)}^l h(x,t)\,h(\xi,t)\,dt \tag{1.8}$$

Let us apply formulas (1.8) to the problem of a plane with a rectilinear skew crack. The WF given in /4/ for the crack tip $h = l$ has the form

$$h(x,l) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{x}{l(l-x)}}$$

Calculations using the second formula of (1.8) yield

$$f(x,\xi;l) = \frac{1}{2\pi}\ln\left|\frac{\sqrt{x(l-\xi)}+\sqrt{\xi(l-x)}}{\sqrt{x(l-\xi)}-\sqrt{\xi(l-x)}}\right|$$

and the first formula of (1.8) leads to the expression for $v(x,\ l)$ already obtained in /2/.

2. Let us consider a strip with a single notch $0 \leqslant x \leqslant l$ emerging at the surface $x = 0$. In this case the WF has the form (H is the width of the strip) /5/

$$h(x,l) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{l-x}}\left[1 + m_1\frac{l-x}{l} + m_2\frac{(l-x)^2}{l^2}\right]$$

$$m_j = A_j + B_j\lambda^2 + C_j\lambda^6,\ j = 1,\ 2;\ 0 \leqslant \lambda \leqslant 0{,}5;\ \lambda = l/H$$

$$A_1 = 0{,}6147,\ B_1 = 17{,}1844,\ C_1 = 8{,}7822$$

$$A_2 = 0{,}2502,\ B_2 = 3{,}2889,\ C_2 = 70{,}0444$$

Let us specify the stress at the crack edges in the form of a polynomial in x (D_k are known function of l)

$$\sigma(x) = \sum_{k=0}^{n} D_k \left(\frac{l-x}{l}\right)^k$$

Then we obtain from (1.7) the displacement of the edge which at the strip surface when $x = 0$, takes the simple form

$$v(0, l) = \frac{4H}{\pi E'} \int_0^{\lambda} (1 + m_1 + m_2) \left[\sum_{k=0}^{n} D_k \left(\frac{1}{2k+1} + \frac{m_1}{2k+3} + \frac{m_2}{2k+5}\right)\right] d\lambda \tag{2.1}$$

The values of $v(0, l)$ for the extension and flexure of the strip are given in /1/. Assuming that $D_0 = \sigma_0$, $D_k = 0$ when $k > 0$ (extension) and $D_0 = \sigma_0(1 - 2\lambda)$, $D_1 = 2\sigma_0\lambda$, $D_k = 0$ when $k > 1$ (flexture), we obtain from (2.1) the values for $v(0, l)$, which are identical with those given in /1/.

3. Let us consider a disc-like crack of radius a in a body acted upon by an axisymmetric load. Repeating the arguments of Sect.1 we obtain for the WF, SIF and the displacement of the crack edge

$$h(r, a) = \frac{1}{2\pi\sqrt{\pi}} \frac{1}{\sqrt{a(a^2 - r^2)}}, \quad K(a) = \frac{2}{\sqrt{\pi a}} \int_0^a \frac{\sigma(r)\, r\, dr}{\sqrt{a^2 - r^2}}$$

$$v(r, a) = \frac{4}{\pi E'} \int_r^a \left[\frac{1}{\sqrt{t^2 - r^2}} \int_0^t \frac{\sigma(\xi)\,\xi\, d\xi}{\sqrt{t^2 - \xi^2}}\right] dt =$$

$$\frac{4}{\pi E'} \left\{\int_0^r \left[\sigma(\xi)\,\xi \int_r^a \frac{dt}{\Delta}\right] d\xi + \int_r^a \left[\sigma(\xi)\,\xi \int_\xi^a \frac{dt}{\Delta}\right] d\xi\right\}$$

$$(\Delta = [(t^2 - r^2)(t^2 - \xi^2)]^{1/2})$$

Note that the last formula can be transformed to that given in /2/, by making the change of variable $t = \xi/\sin\alpha$ in the inner integrals and changing the order of integration.
Using elliptic integrals of the first kind, we reduce $v(r, a)$ to the form

$$v(r, a) = \frac{4}{\pi E'} \int_0^a \sigma(\xi) \frac{\xi}{M} \left[F\left(\frac{m}{M}, \frac{\pi}{2}\right) - F\left(\frac{m}{M}, \arcsin\frac{M}{a}\right)\right] d\xi$$

$$m = \min(\xi, r), \quad M = \max(\xi, r)$$

In particular, for the centre of the crack edge we have

$$v(0, a) = \frac{4}{\pi E'} \int_0^a \sigma(\xi) \arccos\frac{\xi}{a}\, d\xi \tag{3.1}$$

Using the results in /3/, we can show that formula (3.1) can be generalized to the case of a non-axisymmetric load, i.e. $\sigma = \sigma(r, \theta)$. Then the function $\sigma(\xi)$ in (3.1) will be replaced by

$$\sigma_0(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \sigma(\xi, \theta)\, d\theta$$

4. Let us now consider the case of an internal crack bounded by an ellipse L with centre at the origin of coordinates, and the semi-axes a_x, a_y ($a_x < a_y$). The stresses $\sigma_z(x, y)$ are applied at the crack edges. We assume that $\sigma_z(x, y)$ is an even function of x and y. Following /6/, we utilize the RMS values of the SIF at the crack edge, assuming that

$$\bar{K}_{x,y}^2 = \frac{1}{\delta A_{x,y}} K^2\, dA, \quad \delta A_{x,y} = \pi a_{y,x} \delta a_{x,y}$$

with $\bar{K}_x$ and $\bar{K}_y$ depending on the variation in the parameters a_x and a_y respectively.
The WF corresponding to the parameters a_x and a_y, are determined from the formulas /6/

$$h_{x,y}(x, y, a_x, a_y) = \frac{1}{2} \left(\frac{\pi a_{y,x}}{E'} \frac{\partial W}{\partial a_{x,y}}\right)^{-1/2} \frac{\partial v(x, y, a_x, a_y)}{\partial a_{x,y}} \tag{4.1}$$

and are independent of the load. For $\bar{K}_x, \bar{K}_y$ we have the formulas (A is the surface of the ellipse)

$$\bar{K}_{x,y}(a_x, a_y) = 2\iint_A h_{x,y}(\xi, \eta, a_x, a_y)\,\sigma_z(\xi, \eta)\,dA \tag{4.2}$$

$$\bar{K}^2_{x,y} = \frac{E'}{\pi a_{y,x}}\frac{\partial W}{\partial a_{x,y}} \tag{4.3}$$

Using relations (4.1) and (4.3), we shall express $dv/\partial a_x$ and $\partial v/\partial a_y$ in terms of $\bar{K}_x$, $\bar{K}_y$, h_x and h_y. After this we shall integrate the resulting expressions in a_x and a_y respectively, requiring that the condition $v\,|_L = 0$ holds. This will yield

$$v(x, y, a_x, a_y) = \frac{2\pi a_y}{E'}\int_{b_y(x,y)}^{a_x} \bar{K}_x(t, a_y)\,h_x(x, y, t, a_y)\,dt \tag{4.4}$$

or

$$v(x, y, a_x, a_y) = \frac{2\pi a_x}{E'}\int_{b_x(y,x)}^{a_y} \bar{K}_y(a_x, t)\,h_y(x, y, a_x, t)\,dt \tag{4.5}$$

$$b_{x,y}(x, y) = |x|\,[1 - (y/a_{x,y})^2]^{-1/2}$$

where $\bar{K}_x$, $\bar{K}_y$ are obtained from formulas (4.2).

The WF h_x and h_y were obtained in /6/ and transformed to a form suitable for calculation in /7/ (a misprinted expression for h_y in /7/ should be corrected, namely, in the last term the numerator and denominator should be interchanged).

Expressions (4.4) and (4.5) yield the same quantity $v(x, y, a_x, a_y)$. However, all the above formulas were obtained on the assumption that $a_x < a_y$, therefore the lower limit on the right-hand side of (4.5) must satisfy the condition $a_x < b_x(y, x)$, or $x^2 + y^2 > a_x^2$, i.e. formula (4.5) can be used provided that the point x, y lies outside a circle of radius a_x.

5. Let us consider a semi-elliptical surface crack whose plane is perpendicular to the surface of the plate of thickness H. We shall assume that the centre of the ellipse is not displaced by the load and the crack contour remains semi-elliptical, i.e. only the magnitude of the semi-axes a_x and a_y change. The values $\bar{K}_x^*$ and h_x^* corresponding to the parameter a_x, are connected by the relation (A^* is the surface of the semi-ellipse)

$$\bar{K}_x^*(a_x, a_y) = 4\iint_{A^*} h_x^*(\xi, \eta, a_x, a_y)\,\sigma_z(\xi, \eta)\,dA \tag{5.1}$$

The WF h_x^* is obtained by multiplying h_x by the correction function /6, 7/. We must remember here that the correction function used in the present paper differs from that used in /7/ by a factor of 2.

Let us write the formula for the opening of the crack at its centre $\delta = 2v(0, 0, a_x, a_y)$, obtained from formula (4.4) taking into account the expression for the correction function f_n /6, 7/

$$\delta = \frac{24}{\pi E' a_y}\int_0^{a_x}\left\{\frac{D(k)\,f_n(0, t/H)}{tE^2(k)\,[E(k) + D(k)]}\times\right. \tag{5.2}$$

$$\left.\int_0^{a_y} d\eta \int_0^{t\zeta(\eta)}\left[D(k) + \frac{E(k)\,\xi^2}{t^2\alpha}\right]\sqrt{\alpha}\,f_n(x_s, r)\,\sigma_z(\xi, \eta)\,d\xi\right\}dt$$

$$D(k) = E(k) - [F(k) - E(k)]\frac{t^2}{k^2 a_y^2}, \quad k^2 = 1 - \left(\frac{t}{a_y}\right)^2$$

$$\alpha = \zeta^2(\eta) - \left(\frac{\xi}{t}\right)^2, \quad x_s = \frac{\xi}{t\zeta(\eta)}, \quad r = \frac{t\zeta(\eta)}{H}$$

$$\zeta(\eta) = \left[1 - \left(\frac{\eta}{a_y}\right)^2\right]^{1/2}$$

The opening of the crack was calculated using formula (5.2), for the cases of extension and flexure of the plate. The results obtained are compared with the opening δ^* of a crack inclined to the y axis, calculated from the data in /1/ (or from formula (2.1)). Fig.1 shows the dependence of $\lambda = a_x/(2a_y)$ on the parameter $\varkappa = \delta^*/\delta$ characterizing the effect of the form of the crack on its opening when the depth a_x is given, for $H/a_x = 3$ (the dashed line) and $H/a_x = 100$ (the solid line), for two types of stress state (*1* denotes extension and *2* denotes flexure). Using Fig.1 we can determine the opening of a semi-elliptical crack with parameters a_x and a_y by dividing the opening of a skew crack of depth a_x by the corresponding form parameter $\varkappa$. We note that the relative size of the crack and the plate thickness substantially affect the magnitude of $\varkappa$.

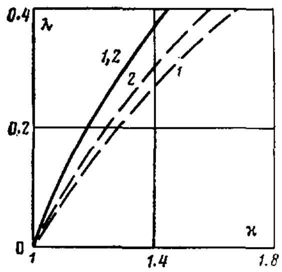

Fig.1

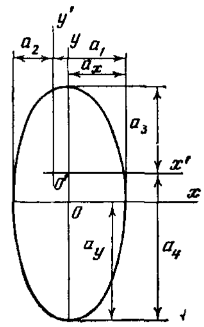

Fig.2

6. We derived in Sect.4 formulas for the opening of an elliptical crack in the case when $\sigma_z(x, y)$ is an even function, i.e. in the case when the centre of the ellipse is not displaced when a load is applied. If this condition does not hold, then following /6/ it is necessary to consider a crack characterized by four parameters a_1, a_2, a_3, a_4 (Fig.2).

The WF corresponding to the parameters a_1, a_2, have the form

$$h_{1,2} = h_x \pm \left[\frac{3a_x}{E(k)+D(k)}\right]^{1/2} \frac{x}{\pi a_x^2 a_y \sqrt{\alpha}}$$

$$a_x = \frac{a_1+a_2}{2}, \quad a_y = \frac{a_3+a_4}{2}, \quad x = x' - \frac{a_1-a_2}{2}, \quad y = y' - \frac{a_3-a_4}{2}$$

The SIF are written thus

$$\bar{K}_{1,2} = 2\iint\limits_A h_{1,2}(x', y', a_1, a_2, a_3, a_4)\sigma_z(x', y')\,dA$$

and the displacement of the crack edge is

$$v = \frac{2\pi a_y}{E'}\int\limits_{b_1+}^{a_1} \bar{K}_1(t, a_2, a_3, a_4)h_1(x', y', t, a_2, a_3, a_4)\,dt, \quad x' \geqslant \frac{a_1-a_2}{2}$$

$$v = \frac{2\pi a_x}{E'}\int\limits_{-a_2}^{b_2-} \bar{K}_2(a_1, t, a_3, a_4)h_2(x', y', a_1, t, a_3, a_4)\,dt, \quad x' \leqslant \frac{a_1-a_2}{2}$$

$$b_j\pm = [2x' \pm a_j(1-\zeta(y))](1+\zeta(y))^{-1}, \quad j = 1, 2$$

It is clear that the quantity v found using the WF h_3, h_4 corresponding to the parameters a_3, a_4 is identical with the one given above.

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